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Classical model of intermediate statistics

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In this work we present a classical kinetic model of intermediate statistics. In the case of Brownian particles we show that the Fermi-Dirac (FD) and Bose-Einstein (BE) distributions can be obtained, just as the Maxwell-Boltzmann (MB) distribution, as steady states of a classical kinetic equation that intrinsically takes into account an exclusion-inclusion principle. In our model the intermediate statistics are obtained as steady states of a system of coupled nonlinear kinetic equations, where the coupling constants are the transmutational potentials $\eta_{\kappa\kappa'}$. We show that, besides the FD-BE intermediate statistics extensively studied from the quantum point of view, we can also study the MB-FD and MB-BE ones. Moreover, our model allows us to treat the three-state mixing FD-MB-BE intermediate statistics. For boson and fermion mixing in a D-dimensional space, we obtain a family of FD-BE intermediate statistics by varying the transmutational potential $\eta_{\rm BF}$. This family contains, as a particular case, when $\eta_{\rm BF}=0$, the quantum statistics recently proposed by L. Wu, Z. Wu, and J. Sun [Phys. Lett. A **170**, 280 (1992)]. When we consider the two-dimensional FD-BE statistics, we derive an analytic expression of the fraction of fermions. When the temperature $T\to\infty$, the system is composed by an equal number of bosons and fermions, regardless of the value of $\eta_{\rm BF}$. On the contrary, when $T\to 0$, $\eta_{\rm BF}$ becomes important and, according to its value, the system can be completely bosonic or fermionic, or composed both by bosons and fermions.

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I. INTRODUCTION

In the last few years, several authors have studied systems of particles which violate the Fermi-Dirac (FD) and Bose-Einstein (BE) statistics. Experimental evidence has been found for these violations and their phenomenology has been studied [1]. Several theoretical approaches have been proposed together with new intermediate or fractional statistics or parastatistics, e.g., the anyonic statistics [2], which is essentially defined in a two-dimensional space, the paron statistics [3], the quon statistics [4], and finally a variant of the quon statistics [5]. All these are quantum statistics, since the concepts of boson and fermion are properly defined only in the framework of quantum physics.

The quon statistics is defined by the quon algebra for the annihilation and creation operators,

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij} \quad .$$

It has been proved that such a Fock-like space exists only if $-1\le q\le 1$. Then the quon algebra is a deformation of the FD ($q=-1$) and BE ($q=1$) algebras as q goes from -1 to 1 on the real axis. When $-1<q<1$ the Fock-like space is the direct sum of all the tensor product powers of the single particle space. The antisymmetric subspace is the Fock space of fermions while the symmetric subspace is the Fock space of bosons, and we have a Maxwell-Boltzmann (MB) statistics [5]. In the case $q=\pm 1$, in addition to the previous commutation relation the quon algebra satisfies also the relation

$$a_i^\dagger a_j^\dagger - q a_j^\dagger a_i^\dagger = 0 \quad .$$

In the variant of quon statistics proposed in Ref. [5], q has been replaced by a linear Hermitian and unitary operator, which has eigenvalues ± 1, obeying the conditions

$$q a_i - a_i q = 0 \ ,$$
$$q a_i^\dagger - a_i^\dagger q = 0 \ .$$

Recently [6] we have shown that the FD distribution can also be obtained in the framework of a classical nonlinear kinetics that takes into account an exclusion principle (EP), when the particles are Brownian. Generalizing the EP and introducing an exclusion-inclusion principle (EIP), it is possible to obtain from a classical equation also the BE distribution [7]. Therefore we conclude that there are three kinds of particles obeying the FD or BE statistics, besides those obeying the MB statistics. These three kinds of particles can be seen as three different states of the same particle.

Introducing the concept of transmutation from a state in which the particle obeys a statistics to another state in which the particle obeys another statistics, it is possible to treat the intermediate statistics by means of a classical kinetic approach. For fermion and boson mixing, we obtain a family of intermediate statistics, which contains as a particular case (when the transmutational potential is zero) the quantum statistics proposed in Ref. [5].

In Sec. II we present the general kinetic equations of the model. In Sec. III we consider the stationary distributions of Brownian particles. In Sec. IV we study the two-state mixing statistics. In Sec. V we consider the fermion and boson mixing, and give particular attention to two-dimensional systems, studying their behavior when the temperature changes. In Sec. VI we discuss

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the three-state mixing statistics. In Sec. VII we consider the intermediate statistics when the transmutational potential is zero. Finally, concluding remarks are given in Sec. VIII.

II. PARTICLE KINETIC EQUATIONS

Let us consider the kinetics of N classical particles in a D-dimensional velocity space. The theory we develop refers to the case of a continuous velocity space, but can be easily extended also to the case of a discrete space. We assume that a particle with velocity $\mathbf{v}$ can stay in s different states obeying s different statistics; we indicate with κ the variable relative to these states. The particle distribution is described by the occupation number indicated by $n_\kappa(t,\mathbf{v})$. A particle in the state κ can diffuse in the velocity space but, at the same time, can change state ($\kappa \to \kappa'$). We call this change of particle state transmutation. The particle kinetics is governed by the continuity equation

$$\frac{\partial n_\kappa(t,\mathbf{v})}{\partial t} + \nabla \mathbf{j}_\kappa(t,\mathbf{v}) + \sum_{\kappa'\neq\kappa} j_{\kappa\leftrightarrow\kappa'}(t,\mathbf{v}) = 0 \quad . \tag{1}$$

$\mathbf{j}_\kappa(t,\mathbf{v})$ represents the particle current in the velocity space, and we assume that it is given by the expression

$$\mathbf{j}_\kappa(t,\mathbf{v}) = -[\mathbf{J}_\kappa(t,\mathbf{v}) + \nabla D_\kappa(t,\mathbf{v})] n_\kappa(t,\mathbf{v})[1+\kappa n_\kappa(t,\mathbf{v})] - D_\kappa(t,\mathbf{v})\, \nabla n_\kappa(t,\mathbf{v}) \ , \tag{2}$$

where $\mathbf{J}_\kappa(t,\mathbf{v})$ and $D_\kappa(t,\mathbf{v})$ are the drift and diffusion coefficients respectively. We note that the current $\mathbf{j}_\kappa(t,\mathbf{v})$ is the sum of two terms. The first one is proportional to $n_\kappa(t,\mathbf{v})[1+\kappa n_\kappa(t,\mathbf{v})]$ and has its origin in the definition of the transition probability of a particle from $\mathbf{v}$ to $\mathbf{v}'$: $\pi_\kappa(t,\mathbf{v}\to\mathbf{v}') \propto n_\kappa(t,\mathbf{v})[1+\kappa n_\kappa(t,\mathbf{v}')]$. We remark that this transition probability depends on the particle population $n_\kappa(t,\mathbf{v})$ of the starting point $\mathbf{v}$ and also on the population $n_\kappa(t,\mathbf{v}')$ of the arrival point $\mathbf{v}'$. If $\kappa = -1$, $\pi_\kappa(t,\mathbf{v}\to\mathbf{v}')$ takes into account the EP. If the arrival site $\mathbf{v}'$ is empty, the transition probability depends only on the population of the starting site. On the other hand, if the arrival site is occupied, the transition is forbidden. The above expression of $\pi_\kappa(t,\mathbf{v}\to\mathbf{v}')$ is the simplest one taking into account the EP during the transition. This form of $\pi_\kappa(t,\mathbf{v}\to\mathbf{v}')$ is valid in the "only individual transition" approximation. In the case of "contemporary transitions," it must be corrected in order to be able to consider the case in which two particles, occupying at the same time two different sites $i\pm 1$, could make a transition to the site i. Of course, due to the EP, one of the two possible transitions is forbidden [6]. If $\kappa = 0$ the transition probability is not affected by the particle population at the arrival site and we have the standard linear Fokker-Planck kinetics. In the case $\kappa = 1$, $\pi_\kappa(t,\mathbf{v}\to\mathbf{v}')$ introduces an inclusion principle. In fact the population at the arrival site stimulates the transition and the transition probability increases linearly with $n_\kappa(t,\mathbf{v}')$. Finally, the second term in the expression of the $\mathbf{j}_\kappa(t,\mathbf{v})$ is the well known Fick current responsible for particle diffusion.

The term $j_{\kappa\leftrightarrow\kappa'}(t,\mathbf{v})$ is the net transmutational current and, according to the exclusion-inclusion principle (EIP), must have the following expression:

$$j_{\kappa\leftrightarrow\kappa'}(t,\mathbf{v}) = g_{\kappa\kappa'}(t,\mathbf{v}) \times\{r_{\kappa\to\kappa'}(t) n_\kappa(t,\mathbf{v})[1+\kappa' n_{\kappa'}(t,\mathbf{v})] - r_{\kappa'\to\kappa}(t) n_{\kappa'}(t,\mathbf{v})[1+\kappa n_\kappa(t,\mathbf{v})]\}. \tag{3}$$

The quantity $r_{\kappa\to\kappa'}(t)$ is the transmutation rate, while the function $g_{\kappa\kappa'}(t,\mathbf{v}) = g_{\kappa'\kappa}(t,\mathbf{v})$ takes into account that the transmutation of a particle from the state κ to the state κ' may depend on the particle velocity. Equations (1)–(3) define unequivocally the process of diffusion and transmutation of the N particles.

III. STATISTICAL DISTRIBUTIONS

In this section we consider the stationary states of the system of particles satisfying the kinetics described in Sec. II. We limit ourselves to the case of Brownian particles, when $t\to\infty$. The statistical distribution or occupational number $n_\kappa(\mathbf{v})$ of the N particles can be seen as the steady state of Eq. (1),

$$n_\kappa(\mathbf{v}) = \lim_{t\to\infty} n_\kappa(t,\mathbf{v}) \quad . \tag{4}$$

The case of Brownian particles is characterized at $t\to\infty$ by

$$\mathbf{J}_\kappa(\infty,\mathbf{v}) = c_\kappa \mathbf{v} \ , \qquad D_\kappa(\infty,\mathbf{v}) = \frac{c_\kappa}{\beta m} \quad , \tag{5}$$

where $\beta = (kT)^{-1}$. The number of particles N_κ in the state κ is given by

$$N_\kappa = \int n_\kappa(\mathbf{v})\, d\tau \quad , \tag{6}$$

where $d\tau = d\tau_1 \cdots d\tau_D$ with $d\tau_i = d(mv_i)dx_i/h$ is the elementary volume in the phase space and h is a constant with the right dimensions which can be identified with the Planck constant. Then, if we call V the volume of the system, we have

$$N_\kappa = V\left(\frac{m}{h}\right)^D \int n_\kappa(\mathbf{v})\, d^D v \quad . \tag{7}$$

The fraction of particles ξ_κ in the state κ is given by

$$\xi_\kappa = \frac{N_\kappa}{N} \quad , \tag{8}$$

and the normalization condition can be written as

$$\sum_\kappa \xi_\kappa = 1 \quad . \tag{9}$$

We can define the average occupation number $\langle n\rangle$ as

$$\langle n\rangle = \sum_\kappa \xi_\kappa n_\kappa \quad . \tag{10}$$

The occupational number n_κ is given as a solution of the differential equation $\mathbf{j}_\kappa(\infty,\mathbf{v}) = \mathbf{0}$. At the same time

n_κ must obey the condition $j_{\kappa\leftrightarrow\kappa'}(\infty,\mathbf{v}) = 0$. When $t \to \infty$, both the particle current in the velocity space and the net transmutation current are zero. After integration of the equation $\mathbf{j}_\kappa(\infty,\mathbf{v}) = \mathbf{0}$, we obtain

$$n_\kappa = \frac{1}{\exp[\beta(E-\mu_\kappa)]-\kappa} \quad , \tag{11}$$

where $E = \frac{1}{2}mv^2$ is the kinetic energy of the particle with velocity $\mathbf{v}$ and the integration constant μ_κ is the chemical potential. For $\kappa = -1$, 0, and 1, Eq. (11) reproduces the FD, MB, and BE statistical distributions, respectively.

IV. TWO-STATE MIXING

In this section we consider the case in which the particles may only be in two different states κ and κ'. Let us write the transmutation rate at $t = \infty$, $r_{\kappa\to\kappa'} = r_{\kappa\to\kappa'}(\infty)$, in the form

$$r_{\kappa\to\kappa'} = \exp(\beta s_{\kappa\to\kappa'}) \quad . \tag{12}$$

We define the transmutational potential $\eta_{\kappa\kappa'}$ as

$$\eta_{\kappa\kappa'} = s_{\kappa\to\kappa'} - s_{\kappa'\to\kappa} \quad , \tag{13}$$

obeying the condition

$$\eta_{\kappa\kappa'} = -\eta_{\kappa'\kappa} \quad . \tag{14}$$

For the stationary state, at $t \to \infty$, the net transmutational current $j_{\kappa\leftrightarrow\kappa'}(t,\mathbf{v})$ must be zero. This fact imposes the following condition on the chemical potentials:

$$\mu_{\kappa'} = \mu_\kappa + \eta_{\kappa\kappa'} \quad . \tag{15}$$

The fraction ξ_κ of particles in the state κ is given by (see the Appendix)

$$\xi_\kappa = a\exp(\beta\mu_\kappa) I_D[\kappa\exp(\beta\mu_\kappa)] \quad , \tag{16}$$

with the constant a given by

$$a = \frac{S_D}{2\rho}\left(\frac{2m}{\beta h^2}\right)^{D/2} \quad , \tag{17}$$

where S_D is the area of a sphere with unitary radius in a D-dimensional space and $\rho = N/V$ is the particle density. The function $I_D(x)$ is defined by means of the integral

$$I_D(x) = \int_0^\infty \frac{t^{D/2-1}}{\exp(t)-x}dt \quad . \tag{18}$$

Using Eq. (10), we can write the average occupational number $\langle n\rangle$ in the form

$$\langle n\rangle = \frac{1 - a\exp[\beta(\mu_\kappa+\eta_{\kappa\kappa'})] I_D\{\kappa'\exp[\beta(\mu_\kappa+\eta_{\kappa\kappa'})]\}}{\exp[\beta(E-\mu_\kappa)]-\kappa} + \frac{a\exp[\beta(\mu_\kappa+\eta_{\kappa\kappa'})] I_D\{\kappa'\exp[\beta(\mu_\kappa+\eta_{\kappa\kappa'})]\}}{\exp[\beta(E-\mu_\kappa-\eta_{\kappa\kappa'})]-\kappa'} \quad . \tag{19}$$

We note that, for $\eta_{\kappa\kappa'} = -\infty$ (no transmutations), all the particles lie in the state κ. In fact from Eq. (19) we have $\bar{n} = n_\kappa$. The chemical potential μ_κ may be calculated from Eq. (9) and is the solution of the following algebraic equation:

$$I_D[\kappa\exp(\beta\mu_\kappa)] + \exp(\beta\eta_{\kappa\kappa'}) \times I_D\{\kappa'\exp[\beta(\mu_\kappa+\eta_{\kappa\kappa'})]\} = \frac{1}{a}\exp(-\beta\mu_\kappa) \quad . \tag{20}$$

Equation (19) defines a statistics that is different from the FD, MB, and BE ones. For $\kappa = -1, \kappa' = 1$ (or $\kappa = 1, \kappa' = -1$) we have a FD-BE intermediate statistics; for $\kappa = -1, \kappa' = 0$ (or $\kappa = 0, \kappa' = -1$) we have a FD-MB intermediate statistics, and finally for $\kappa = 1, \kappa' = 0$ (or $\kappa = 0, \kappa' = 1$) we have a BE-MB intermediate statistics.

V. BOSON-FERMION MIXING

In this section we consider the FD-BE intermediate statistics that describe a mixing of bosons and fermions. The fraction of fermions is

$$\xi_F = a\exp[\beta(\mu_B+\eta_{\rm BF})] I_D\{-\exp[\beta(\mu_B+\eta_{\rm BF})]\} \quad , \tag{21}$$

where μ_B may be determined as solution of Eq. (20) with $\kappa = 1, \kappa' = -1$ and $\mu_\kappa = \mu_B, \eta_{\kappa\kappa'} = \eta_{\rm BF}$. The average occupational number is given by

$$\langle n\rangle = \frac{1}{\exp[\beta(E-\mu_B)]-1} - \xi_F \frac{2 - [1-\exp(-\beta\eta_{\rm BF})]\exp[\beta(E-\mu_B)]}{\exp(-\beta\eta_{\rm BF})\exp[2\beta(E-\mu_B)] + [1-\exp(-\beta\eta_{\rm BF})]\exp[\beta(E-\mu_B)] - 1} \quad . \tag{22}$$

Equation (22) gives the average occupational number as a correction of the occupational number of a system of bosons. The correction is due to the presence of a fraction ξ_F of fermions. Obviously $\langle n\rangle$ can be written as a correction of the occupational number of a system of fermions, due to the presence of a fraction ξ_B of bosons.

Let us examine the statistics of a two-dimensional boson-fermion mixing. In this case the integral $I_2(x)$ of Eq. (18) can be easily calculated (see the Appendix). If we introduce the energy ν,

$$\nu = \frac{h^2\rho}{2\pi m} \quad , \tag{23}$$

we have for the chemical potential μ_B

$$\exp(\beta\mu_B) = \frac{\exp(\beta\nu) - 1}{\exp(\beta\nu) + \exp(\beta\eta_{BF})} \quad , \tag{24}$$

and for the fraction of fermions

$$\xi_F = \frac{1}{\beta\nu} \ln \frac{[1 + \exp(\beta\eta_{BF})] \exp(\beta\nu)}{\exp(\beta\eta_{BF}) + \exp(\beta\nu)} \quad . \tag{25}$$

We observe that, for $\eta_{BF} = \nu/2$, ξ_F is independent of the temperature $T = 1/(k\beta)$ and assumes the value $\xi_F = 1/2$. From Eq. (25) for $T \to \infty$ we have

$$\xi_F \approx \frac{1}{2} + \frac{1}{4}\left(\frac{\eta_{BF}}{\nu} - \frac{1}{2}\right) \frac{\nu}{kT} \quad . \tag{26}$$

Then the system is composed of an equal number of bosons and fermions. For $T \to 0$ from Eq. (25) we obtain

$$\xi_F = \begin{cases} 0, & \eta_{BF} \leq 0 \\ \eta_{BF}/\nu, & 0 < \eta_{BF} < \nu \\ 1, & \eta_{BF} \geq \nu. \end{cases} \tag{27}$$

We remark that the transmutational potential η_{BF} is very important because the fraction of fermions ξ_F at $T = 0$ depends on its value. For $\eta_{BF} \leq 0$ all the particles are bosons, while for $\eta_{BF} \geq \nu$ all the particles are fermions. Instead, when $0 < \eta_{BF} < \nu$ the fermion fraction is η_{BF}/ν (and then the boson fraction is $1 - \eta_{BF}/\nu$).

VI. THREE-STATE MIXING

In this section we consider particles that can be in three different states $\kappa = -1, 0, 1$. The occupational number of the state κ is given for $t \to \infty$ by Eq. (11) and, as a consequence, we do not have particle current in the velocity space. If we impose the condition that for $t \to \infty$ also the three net transmutational currents be zero, we obtain two conditions on the chemical potentials μ_κ,

$$\mu_M = \mu_B + \eta_{BM} \quad , \tag{28}$$

$$\mu_F = \mu_B + \eta_{BF} \quad , \tag{29}$$

and one condition on the transmutational potentials $\eta_{\kappa\kappa'}$,

$$\eta_{BF} = \eta_{BM} + \eta_{MF} \quad . \tag{30}$$

The conditions given by Eqs. (28) and (29) allow the evaluation of μ_M and μ_F as a function of μ_B. The condition given by Eq. (30) requires that the transmutational potential of the two-step, indirect transmutation $B \to M \to F$ be equal to the transmutational potential of the one-step, direct transmutation $B \to F$. For the average occupational number we obtain the following expression:

$$\langle n \rangle = \frac{1}{\exp[\beta(E - \mu_{\kappa_1})] - \kappa_1} + \sum_{j=2,3} \xi_{\kappa_j} \frac{\kappa_j - \kappa_1 + [1 - \exp(-\beta\eta_{\kappa_1\kappa_j})] \exp[\beta(E - \mu_{\kappa_1})]}{\{\exp[\beta(E - \mu_{\kappa_1})] - \kappa_1\}\{\exp[\beta(E - \mu_{\kappa_j} - \eta_{\kappa_1\kappa_j})] - \kappa_j\}} \quad . \tag{31}$$

μ_{κ_1} results as the solution of the equation

$$I_D[\kappa_1 \exp(\beta\mu_{\kappa_1})] + \sum_{j=2,3} \exp(\beta\eta_{\kappa_1\kappa_j}) I_D\{\kappa_j \exp[\beta(\mu_{\kappa_1} + \eta_{\kappa_1\kappa_j})]\} = \frac{1}{a} \exp(-\beta\mu_{\kappa_1}) \quad , \tag{32}$$

and ξ_{κ_j} is given by

$$\xi_{\kappa_j} = a \exp[\beta(\mu_{\kappa_1} + \eta_{\kappa_1\kappa_j})] \times I_D\{\kappa_j \exp[\beta(\mu_{\kappa_1} + \eta_{\kappa_1\kappa_j})]\} \quad . \tag{33}$$

Equations (31) and (32) reproduce the two-state mixing statistics if we let the index j take only the value $j = 2$. We note, from Eq. (31), that the average occupational number is given as a correction of the occupational number when the system is in the state κ_1. The correction is given by the two terms of the sum and is due to the presence of the fractions ξ_{κ_2} and ξ_{κ_3} of the particles that lie in the states κ_2 and κ_3.

VII. SYMMETRIC TRANSMUTATION

In Sec. V we have considered the mixing of bosons and fermions. Equation (22) gives the average occupational number of the system and defines a family of statistics, varying the transmutational potential η_{BF}. In the particular case in which $\eta_{BF} = 0$, i.e., when the transmutational rates $r_{\kappa\to\kappa'}$ and $r_{\kappa'\to\kappa}$ are equal, we have $\mu_F = \mu_B = \mu$ and the average occupational number assumes the following form:

$$\langle n \rangle = \frac{1}{\exp[\beta(E - \mu)] - 1} - \frac{2\xi_F}{\exp[2\beta(E - \mu)] - 1} \quad . \tag{34}$$

The intermediate statistics given by Eq. (34) has been recently obtained by Wu, Wu, and Sun [5] in a quantum context and the particles obeying this statistics are a particular type of quons.

Finally in the case of a symmetric transmutation for the three-state mixing statistics treated in Sec. VI we obtain

$$\langle n \rangle = \frac{1}{\exp[\beta(E - \mu)] - 1} - \frac{2\xi_F}{\exp[2\beta(E - \mu)] - 1} - \frac{\xi_M}{\exp[2\beta(E - \mu)] - \exp[\beta(E - \mu)]} \quad . \tag{35}$$

VIII. CONCLUSIONS

We have presented a classical kinetic model for intermediate statistics. We have shown that the steady states of a nonlinear kinetics that takes into account the EIP are given by the FD, MB, and BE distributions. In our model the intermediate statistics are obtained as steady states of a system of coupled kinetic equations where the coupling constants are the transmutational potentials $\eta_{\kappa\kappa'}$. Besides the FD-BE intermediate statistics, extensively studied from the quantum point of view, we have studied also the MB-FD and MB-BE ones. Moreover, our model allows us to treat the three-states mixing FD-MB-BE statistics. For the boson-fermion mixing in a space of arbitrary dimension D, we obtain a family of FD-BE intermediate statistics, varying the transmutational potential η_{BF}. This family contains as a particular case when $\eta_{BF} = 0$ the quantum statistics proposed by Wu, Wu, and Sun [5].

In the case of the two-dimensional FD-BE intermediate statistics, it is possible to calculate analytically the fermion fraction ξ_F (and then the boson fraction) as a function of the temperature T, the transmutation potential η_{BF} and the number N of particles. For $T \to \infty$ independent of η_{BF} and N, the system is always composed of an equal number of fermions and bosons. On the contrary, when $T \to 0$, the system composition depends strongly on η_{BF}. For $\eta_{BF} \leq 0$ all the particles are bosons, while for $\eta_{BF} \geq \nu$ all the particles are fermions. When $0 < \eta_{BF} < \nu$ the fermion fraction is η_{BF}/ν (and then the boson fraction is $1 - \eta_{BF}/\nu$). We note that, when $\eta_{BF} = \nu/2$, independent of the temperature, the system is formed by an equal number of fermions and bosons.

A study of the FD-BE statistics for $D \neq 2$ and the FD-MB, BE-MB, FD-MB-BE statistics for different values of D, would be very important, because the fraction of the different kinds of particles depends on D.

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APPENDIX

We start from the definition of the fraction of particles in the state κ given by Eqs. (7) and (8),

$$\xi_\kappa = \frac{1}{\rho}\left(\frac{m}{h}\right)^D \int \frac{d^D v}{\exp[\beta(E-\mu_\kappa)] - \kappa} \quad , \tag{A1}$$

where $\rho = N/V$ is the particle density. After defining the dimensionless velocity u_i

$$u_i = \sqrt{\frac{m\beta}{2}}\, v_i \quad , \tag{A2}$$

we have

$$\xi_\kappa = \frac{1}{\rho}\left(\frac{2m}{\beta h^2}\right)^{D/2} \exp(\beta\mu_\kappa) \times \int \frac{d^D u}{\exp(u_1^2 + \ldots + u_D^2) - \kappa \exp(\beta\mu_\kappa)} \quad . \tag{A3}$$

For the calculation of the integral we observe that if we call u the modulus of the vector $\mathbf{u} = (u_1, \ldots, u_D)$, we have

$$d^D u = du_1 \cdots du_D = S_D u^{D-1} du \quad , \tag{A4}$$

where S_D is the area of a sphere with unitary radius in a D-dimensional space ($S_1 = 2$, $S_2 = 2\pi$, $S_3 = 4\pi$, $S_4 = 2\pi^2$, $S_5 = \frac{8}{3}\pi^2$, $S_6 = \pi^3$, $S_7 = \frac{16}{15}\pi^3$, ...). Then we have

$$\xi_\kappa = \frac{1}{\rho}\left(\frac{2m}{\beta h^2}\right)^{D/2} \exp(\beta\mu_\kappa) S_D \times \int_0^\infty \frac{u^{D-1} du}{\exp(u^2) - \kappa \exp(\beta\mu_\kappa)} \quad . \tag{A5}$$

Let us make the substitution of the integration variable

$$u \to t = u^2 \quad , \tag{A6}$$

and introduce the function

$$I_D(x) = \int_0^\infty \frac{t^{D/2-1} dt}{\exp(t) - x} \tag{A7}$$

and the constant

$$a = \frac{S_D}{2\rho}\left(\frac{2m}{\beta h^2}\right)^{D/2} \quad . \tag{A8}$$

We obtain the following expression for the fraction of particles in the state κ:

$$\xi_\kappa = a \exp(\beta\mu_\kappa) I_D[\kappa \exp(\beta\mu_\kappa)] \quad . \tag{A9}$$

The value of the function $I_D(x)$ for $x = 0$ is

$$I_D(0) = \int_0^\infty t^{D/2-1} \exp(-t) dt = \Gamma\left(\frac{D}{2}\right) \quad . \tag{A10}$$

$I_D(-x)$ can be calculated starting from $I_D(x)$ and $I_D(x^2)$ according to

$$I_D(-x) = I_D(x) - 2^{1-D/2} x I_D(x^2) \quad . \tag{A11}$$

In the case of a two-dimensional space, we can introduce the energy

$$\nu = \frac{h^2\rho}{2\pi m} \quad , \tag{A12}$$

and we have for the constant a the following simple expression:

$$a = \frac{1}{\beta\nu} \quad . \tag{A13}$$

Finally the function $I_2(x)$ can be easily calculated,

$$I_2(x) = \frac{1}{x} \ln \frac{1}{|1-x|} \quad ; \quad x \neq 0 \quad , \tag{A14}$$

$$I_2(0) = 1 \quad . \tag{A15}$$

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